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Painlevé Test for Long-Wave, Short-Wave Interaction Equation. II

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In a previous communication we discussed the integrability of a long-wave, short-wave interaction equation for very restricted values of the parameters in the framework of the theory laid down by Weiss *et al.* Here we proceed with the analysis in the sense of Weiss *et al.*, considering values of parameters other than those we used previously. We observe that for the combination of parameters for which a Lax pair was obtained by another approach (Newell), the equations considered pass the Painlevé test for integrability in the sense of Weiss *et al.* We discuss several other combinations of parameters that do not pass the test. For these cases no Lax Pair was reported by Newell.

1. INTRODUCTION

The question of the integrability of nonlinear partial differential equations occupies a central role in the study of solitons and their properties. For partial differential equations, which are infinite dimensional, integrability [sometimes called "complete integrability" (Tabor and Gibbon, 1986)] is shown by the existence of an infinite number of integrals in involution (Gibbon *et al.*, 1985). Zakharov and Faddeev (1972) demonstrated a procedure for finding integrals via inverse-scattering transform for the KdV equation [also see Flaschka and Newell (1975) in the context]. The consequences of this work are such that once an isospectral problem has been found for a pde, then a direct connection can be established between the existence of an infinity of conserved quantities and the existence of multiple solitons. The multiple-soliton solutions can be found by the inverse scattering transform method (Ablowitz and Segur, 1981) pioneered by Kruskal and co-workers or by other methods, such as that developed by Hirota (1971, 1980). The constructive approach of the method of differential forms

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(Wahlquist and Estabrook, 1975) ends up providing a complete solution of the problem in that it determines the inverse scattering transform (IST). In the light of these observations one may say that a nonlinear partial differential equation solvable by IST is integrable.

In an entirely different development Ablowitz *et al.* (1978; Ablowitz, 1980) introduced the idea of the Painlevé property in connection with the integrability of nonlinear partial differential equations. They conjectured that every nonlinear ordinary differential equation obtained by an exact reduction (e.g., through similarity transformation) of an integrable nonlinear partial differential equation (i.e., solvable by the inverse scattering method) has the Painlevé property, namely its general solution can have no movable singular points other than poles. Furthermore, they proposed a consequence of the conjecture to be an explicit test of whether or not a given nonlinear partial differential equation may be of IST class. Partial proofs of this conjecture have been given by Ablowitz *et al.* (1978, 1980) and McLeod and Olver (1983). Weiss *et al.* (1983) extended the idea of Ablowitz *et al.* (1978; Ablowitz, 1980) directly to nonlinear partial differential equations instead of going indirectly via ordinary differential equations found by similarity transformations, etc. According to them, a nonlinear partial differential equation is said to possess the Painlevé property if the solutions of the nonlinear partial differential equation are "single-valued" in the neighborhood of a movable singularity manifold. In addition to the references already cited one can consult the review of Tabor and Gibbon (1986) or Steeb *et al.* (1985) for further details.

According to Weiss (1985), the Painlevé test is as follows:

If the singularity manifold is determined by

$$\phi(Z_1, Z_2, \ldots, Z_n) = 0 \tag{1}$$

and $u = u(Z_1, \ldots, Z_n)$ is a solution of the pde, then we require that

$$u = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^j \tag{2}$$

where $u_0 \neq 0$, and $\phi = \phi(Z_1, \ldots, Z_n)$ and $u_j = u_j(Z_1, \ldots, Z_n)$ are analytic functions of (Z_j) in the neighborhood of the manifold (1), and α (the leading order exponent) is a (negative) rational number. The requirement that the manifold (1) be noncharacteristic (for the pde) ensures that the expansion (2) will be well defined, in the sense of the Cauchy-Kovalevskaya theorem (Courant and Hilbert, 1962). Substitution of (2) into the pde determines that value(s) of α, and defines the recursion relations for u_j, $j = 0, 1, 2, \ldots$. When the expansion (2) is well defined and contains the maximum number of arbitrary functions allowed at the "resonances" (Weiss *et al.*, 1983; Ablowitz, 1980; Yoshida, 1983a,b), the pde is said to possess the Painlevé property and is conjectured to be integrable. Informally, the resonances are the values of j for which u_j are not "fixed" by the recursion relations (i.e., are arbitrary).

The success achieved by the formalism of Weiss *et al.* (1983; Weiss, 1984a,b, 1986) is noteworthy.

However, the rigor of both the approaches due to Ablowitz *et al.* (1978; Ablowitz, 1980) and due to Weiss *et al.* (1983) have been questioned by several authors (Clarkson, 1985, 1986; Ward, 1984, 1985). Ward (1984, 1985) suggested an approach that seems to be more rigorous than the above two approaches and at the same time too complicated to be applied in actual situations. Actually, in the work of Ward (1984, 1985) it could not be observed how one might determine whether the KdV equation possess the Painlevé property in the sense due to Ward.

In a recent communication we have applied the Painlevé test in the sense of Weiss *et al.* (1983; Weiss, 1985a,b) to the equations governing the long-wave, short-wave interaction:

$$A_t = 2S(BB^*)_x \tag{3a}$$

$$B_t - 2iB_{xx} = K_2 A_x B - K_3 A B_x + iK_4 A^2 B - 2iSB^2 B^* \tag{3b}$$

where K_2, K_3, K_4, and S are arbitrary constants, of which K_2 and K_3 may be complex.

In that paper our analysis was restricted to some very special values of the parameters given by $K_2 = 0$, $K_3 = 0$, $K_4 = 0$.

In the present paper we apply the Painlevé test in the sense of Weiss *et al.* (1983; Weiss, 1985a,b) to the system (3) for K_2, K_3, K_4 not simultaneously zero. The Painlevé test in the sense of Weiss *et al.* (1983; Weiss, 1985a,b) may be divided into three main steps after the substitution of (2) in the differential equations concerned:

1. Make the leading order analysis [where one gets all possible α and u_0 in (1)].
2. Define the recursion relations for u_j for the leading orders obtained in step 1 and determine the resonance positions (those values of j for which the recursion relations are not defined).
3. Check whether one actually gets the introduction of arbitrariness at the resonance positions; this is ascertained only when the compatibility conditions that arise at the resonance positions $j = r$ are identically satisfied for all the results at $j < r$.

We attempted to proceed through the above steps with general values K_2, K_3, K_4 and S in (3) so that the analysis itself could impose restrictions on the parameters for the system (3) passing the Painlevé test in the sense of Weiss *et al.* (1983; Weiss, 1985a,b). We could do this for steps 1 and 2, which imposed some conditions on K_2, K_3, and K_4. However, unless one knows the resonance positions exactly, the checking of arbitrariness at the resonance positions cannot be done. Hence, for step 3 this general procedure could not be continued and we restricted ourselves to some particular values (or combinations) of K_2, K_3, and K_4 (which obey the conditions already imposed steps in 1 and 2. These particular cases included the physically interesting situation discussed by Newell (1979) for which he obtained a Lax pair. It is interesting to note that the system (3) passed the Painlevé test (Weiss *et al.*, 1983, Weiss, 1985a,b) for this situation. For the other particular values of K_2, K_3, and K_4 not discussed by Newell (1979) the system (3) did not pass the test. Actually it happens that in these situations at least one of the conditions for passing the test [i.e., the number of arbitrary functions in (2) should contain the maximum number of arbitrary functions allowed at the resonances] is violated.

The equations discussed by Newell (1979) may seem to differ from those discussed by us. However, they are actually equivalent to (3) under scaling transformation, as shown in the following.

The equations discussed by Newell (1979) are

$$A_t = 2S(BB^*)_x, \qquad S = \pm 1 \tag{4a}$$

$$B_t - iB_{xx} = -K_3 A_x B - \bar{K}_3 A B_x + iK_4 A^2 B - 2iSB^2 B^* \tag{4b}$$

Under the scaling transformation

$$\begin{gathered} \frac{\partial}{\partial x} \to \frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial t} \to \frac{1}{2}\frac{\partial}{\partial t}, \qquad A \to A, \qquad B \to B, \qquad B^* \to B^* \\ K_3 \to -\frac{K_2}{2}, \qquad \bar{K}_3 \to \frac{K_3}{2}, \qquad K_4 \to \frac{K_4}{2}, \qquad S \to \frac{S}{2} \end{gathered} \tag{5}$$

the system (4) reduces to the system (3).

Newell (1979) obtained a Lax pair for (4) for

$$K_3 = 1, \qquad \bar{K}_3 = 0, \qquad K_4 = 1 \tag{6}$$

Newell (1979) also obtained a Lax pair for (4) in three other situations, for one of which the equations are related to the equations with (6) under the transformation $B \to B \exp(-\frac{5}{3} i \int A\, dx)$ and the other two make (4) separable when (4) transform to linear equations. Thus, we have concentrated our investigation on (4) with (6), which under (5) is equivalent to (3) with

$$K_2 = -2, \qquad K_3 = 0, \qquad K_4 = 2 \tag{7}$$

2. LEADING ORDER ANALYSIS

Equations (3) can be rewritten as

$$\begin{aligned} A_t &= 2S(BC)_x \\ B_t - 2iB_{xx} &= K_2 A_x B - K_3 A B_x + iK_4 A^2 B - 2iSB^2 C \\ C_t + 2iC_{xx} &= \bar{K}_2 A_x C - \bar{K}_3 A C_x - iK_4 A^2 C + 2iSC^2 B \end{aligned} \tag{8}$$

For the Painlevé test we set

$$A = \phi^\alpha \sum_{j=0}^{\infty} a_j \phi^j; \qquad B = \phi^\beta \sum_{j=0}^{\infty} b_j \phi^j; \qquad C = \phi^\gamma \sum_{j=0}^{\infty} c_j \phi^j \tag{9}$$

In leading order analysis we assume

$$A \sim a_0 \phi^\alpha, \qquad B \sim b_0 \phi^\beta, \qquad C \sim c_0 \phi^\gamma$$

where all a_i, b_i, c_i are functions of t and $\phi = x - f(t)$, a prescription given by Kruskal (first used in 1982), in order to simplify the original theory of Weiss *et al.* [see the Appendix of Weiss *et al.* (1983) and the introduction of the review by Tabor and Gibbon (1986)]. By inspection we observe that

$$\alpha = -1 \tag{10a}$$

$$\beta + \gamma = -1 \tag{10b}$$

$$a_0 \dot{f} = -2S b_0 c_0 \tag{10c}$$

$$-2i\beta(\beta - 1) = -(K_2 + K_3 \beta) a_0 + iK_4 a_0^2 \tag{10d}$$

$$+2i\gamma(\gamma - 1) = -(\bar{K}_2 + K_3 \gamma) a_0 - iK_4 a_0^2 \tag{10e}$$

whence the dominant terms are

$$\begin{aligned} A_t &= 2S(BC)_x \\ -2iB_{xx} &= K_2 A_x B - K_3 A B_x + iK_4 A^2 B \\ +2iC_{xx} &= \bar{K}_2 A_x C - \bar{K}_3 A C_x - iK_4 A^2 C \end{aligned}$$

In the general case the constants K_2 and K_3 are complex. Here $\bar{K}_2$ and $\bar{K}_3$ are the complex conjugates of K_2 and K_3, and K_4 and S are real constants.

From equations (10) we observe that β and γ are not uniquely determined. So we set

$$\beta = -p \qquad (p \text{ real})$$

Then

$$\gamma = p - 1$$

and equations (10) yield

$$\begin{aligned} a_0\dot{f} &= -2Sb_0c_0 \\ -2ip(p+1) &= -(K_2-K_3p)a_0+iK_4a_0^2 \\ 2i(p-1)(p-2) &= -[\bar{K}_2+\bar{K}_3(p-1)]a_0-iK_4a_0^2 \end{aligned} \tag{11}$$

Here we have two equations quadratic in a_0 and p. To be consistent, we must have:

(i) Either both the roots for a_0 are the same, whence we get the condition

$$\frac{-p(p+1)}{(p-1)(p-2)}=\frac{K_2-K_3p}{\bar{K}_2+\bar{K}_3(p-1)}=-1 \tag{12}$$

The ratio for p yields $p=\frac{1}{2}$ and the second ratio gives

$$2(K_2+\bar{K}_2)=(K_3+\bar{K}_3) \tag{13}$$

and the single equation for a_0 reads

$$2iK_4a_0^2-(2K_2-K_3)a_0+3i=0 \tag{14}$$

Note that the case $K_4=0$, $p=\frac{1}{2}$ is included in this case with $a_0=6i/(2K_2-K_3)$.

(ii) Or, when the roots are different, then

$$K_2+\bar{K}_2+K_3(p-1)-K_3p\neq 0 \tag{15}$$

and we get

$$a_0=\frac{4i(2p-1)}{K_2+\bar{K}_2+\bar{K}_3(p-1)-K_3p} \tag{16}$$

On the other hand, we can also solve for a_0^2 and obtain

$$a_0^2=\frac{-2\{p(p+1)[\bar{K}_2+\bar{K}_3(p-1)]+[(p-1)(p-2)(K_2-K_3p)]\}}{K_4[K_2+\bar{K}_2+(p-1)\bar{K}_3-K_3p]} \tag{17}$$

From (16) and (17) we get a connection between p and K_i, $\bar{K}_i$, given as

$$\begin{aligned} 8(2p-1)^2K_4 &= \{p(p+1)[\bar{K}_2+\bar{K}_3(p-1)] \\ &\quad +(p-1)(p-2)(K_2-K_3p)\}[\bar{K}_2+\bar{K}_3(p-q)-K_3p] \end{aligned} \tag{18}$$

It may be noted that $p=\frac{1}{2}$ is not allowed in equation (16).

As an example of the above general discussion, we observe what happens with the particular values of K_i and $\bar{K}_i$ in (8), which lead to a situation equivalent to that for which Newell (1979) obtained a Lax pair.

Here we have from (7)

$$K_2=-2, \qquad K_3=0, \qquad K_4=2, \qquad \bar{K}_2=-2, \qquad \bar{K}_3=0$$

These do not satisfy (12) and hence we are left with (15)-(18).

Using these in (18), one gets $p=0$, 1.

Then from (16) one has

$$\text{(i) } p=0,\ a_0=i \qquad \text{or} \qquad \text{(ii) } p=1,\ a_0=-i \tag{19}$$

3. RECURSION RELATION AND RESONANCES

Let us set

$$A=\sum_{j=0}^{\infty} a_j(t)\phi^{j-1}, \qquad B=\sum_{j=0}^{\infty} b_j(t)\phi^{j-p}, \qquad C=\sum_{j=0}^{\infty} c_j(t)\phi^{j+p-1}$$

If we substitute these in equations (8) and equate coefficients of ϕ, we get

$$\begin{bmatrix} (n-1)\dot{f} & 2Sc_0(n-1) & 2Sb_0(n-1) \\ \begin{matrix}[K_2(n-1)+pK_3 \\ +2iK_4a_0]b_0\end{matrix} & \begin{matrix}2i(n-p)(n-p-1) \\ -K_2a_0+iK_4a_0^2 \\ -K_3a_0(n-p)\end{matrix} & 0 \\ \begin{matrix}[\bar{K}_2(n-1)-(p-1)\bar{K}_3 \\ -2iK_4a_0]c_0\end{matrix} & 0 & \begin{matrix}-2i(n+p-1) \\ \times(n+p-2) \\ -\bar{K}_2a_0-iK_4a_0^2 \\ -\bar{K}_3a_0(n+p-1)\end{matrix} \end{bmatrix} \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{20}$$

with

$$X=\dot{a}_{n-1}-2S\sum_{q=1}^{n-1} b_{n-q}c_q(n-1)$$

$$\begin{aligned} Y=\dot{b}_{n-2}-b_{n-1}(n-p-1)\dot{f}-K_2\sum_{q=1}^{n-1} a_{n-q}b_q(n-q-1) \\ +K_3\sum_{q=1}^{n-1} a_{n-q}b_q(q-p)-iK_4\sum_{\substack{d=0 \\ d+q>0}}^{n-1}\sum_{q=0}^{n-1} a_{n-q-d}a_db_q \\ +2iS\sum_{d=0}^{n-1}\sum_{q=0}^{n-1} b_{n-q-d-1}b_dC_q \end{aligned} \tag{21}$$

$$Z = \dot{C}_{n-2} - C_{n-1}(n+p-2)\dot{f} - \bar{K}_2 \sum_{q=1}^{n-1} a_{n-q}C_q(n-q-1)$$

$$+ \bar{K}_3 \sum_{q=1}^{n-1} a_{n-q}C_q(q+p-1) + iK_4 \sum_{\substack{d=0 \\ d+q>0}}^{n-1} \sum_{q=0}^{n-1} a_{n-q-d}a_dC_d$$

$$-2iS \sum_{d=0}^{n-1} \sum_{q=0}^{n-1} C_{n-q-d-1}C_db_q \tag{22}$$

The matrix occurring on the left-hand side of (20) is called the system matrix (SM) and the resonance positions are determined by the condition that at these the determinant of the system matrix vanishes. We obtain after some simplification,

$$\det[\mathrm{SM}] = 4\dot{f}(r+1)r(r-1)(r-\sigma)(r-\tau)$$

So that the resonances are

$$r = -1, 0, 1, \sigma, \tau$$

where σ, τ are the roots of the quadratic equation

$$4r^2 + \sigma' r + \sigma'' = 0 \tag{23}$$

where σ' and σ'' are given by (25) and (26); here:

(i) $r = -1$ corresponds to the arbitrariness of $\phi(x, t) = x - f(t)$.
(ii) $r = 0$ corresponds to the arbitrariness of one of b_0 and c_0 in (10).

Now, since a resonance position is meaningful if and only if it occurs at an integer point, we demand

$$\begin{aligned} \sigma + \tau &= \text{integer (positive)} \\ \tau\sigma &= \text{integer (positive)} \end{aligned} \tag{24}$$

which lead to the conditions

$$\sigma + \tau = \text{integer (positive)}$$

$$= -\tfrac{1}{4}(-20 + 2iK_3a_0 - 2i\bar{K}_3a_0 - 2iK_2a_0 + 2ia_0\bar{K}_2) \tag{25}$$

$$\sigma\tau = \tfrac{1}{4}[(-32p^2 + 32p + 16) + (6p-8)iK_3a_0$$

$$-(6p+2)(i\bar{K}_3a_0) - (4p-6)iK_2a_0$$

$$-(4p+2)i\bar{K}_2a_0 + a_0^2(K_3\bar{K}_3 - K_2\bar{K}_3 - K_3\bar{K}_2)] \tag{26}$$

However, it may be noted that when σ and τ are negative, the expansion (9) will have a probability of representing the special solution (not obtainable from the general solution) and not the general solution in the neighborhood of the singularity manifold $\phi(x, t)=0$. And, when σ, τ, etc., are fractions, one may need to have the "weak Painlevé" expansions in analogy to those for ODES (Ramani *et al.*, 1982; Ranada *et al.*, 1985; Graham *et al.*, 1985) or those for PDES (Weiss, 1986). We do not go into the details here.

From (25) one has the observation that

$$a_0 = \frac{2i[(\sigma+\tau)-5]}{(K_3-\bar{K}_3)-(K_2-\bar{K}_2)} \tag{27}$$

so that for a_0 to be nonzero we must have simultaneously

$$\sigma+\tau=5 \tag{28a}$$

$$K_3-\bar{K}_3=K_2-\bar{K}_2 \tag{28b}$$

For combinations other than (28b) one must have $\sigma+\tau\neq 5$.

It should be mentioned that these conditions all hold at the values given by (12)-(18).

3.1. Results for Leading Orders Defined in (12)–(14)

Here we have

$$\sigma=3+\tfrac{1}{4}i[2(K_2-\bar{K}_2)-(K_3-\bar{K}_3)]a_0$$
$$\tau=2-\tfrac{1}{4}i(K_3-\bar{K}_3)a_0$$

from which we can deduce

$$\begin{aligned} i(K_3-\bar{K}_3)a_0 &= 4(2-\tau) \\ i(K_2-\bar{K}_2)a_0 &= 2(\sigma-\tau)-2 \end{aligned} \tag{30}$$

From equations (28) we can have the following possible choices of (30):

a. $(\tau=2, \sigma=3)$, $(\tau=3, \sigma=2)$: For $\tau=2$, $\sigma=3$, we get from (30) $K_3=\bar{K}_3$, $K_2=\bar{K}_2$. On the other hand, for $\tau=3$, $\sigma=2$, we get

$$a_0=4i/(K_2-\bar{K}_2) \tag{31}$$

along with $K_3-\bar{K}_3=K_2-\bar{K}_2$, but $K_3\neq\bar{K}_3$, $K_2\neq\bar{K}_2$. Now from (28b), (13), and (14) we are led to

$$\begin{aligned} \bar{K}_3 &= \tfrac{1}{2}(K_2+3\bar{K}_2) \\ K_3 &= \tfrac{1}{2}(3K_2+\bar{K}_2) \\ K_4 &= \tfrac{1}{32}(K_2-\bar{K}_2)^2 \end{aligned} \tag{32}$$

b. $(\tau=0, \sigma=5)$, $(\tau=5, \sigma=0)$: As before, for $\tau=0$, $\sigma=5$, we get

$$K_3=\tfrac{1}{2}(3K_2+\bar{K}_2), \qquad \bar{K}_3=\tfrac{1}{2}(K_2+3\bar{K}_2)$$
$$a_0=-8i/(K_2-\bar{K}_2), \qquad K_4=\tfrac{7}{128}(K_2-\bar{K}_2)^2$$

For $\tau=5$, $\sigma=0$, the same set of values for K_3, $\bar{K}_3$ is obtained with

$$a_0=12i/(K_3-\bar{K}_3), \qquad K_4=-\tfrac{3}{288}(K_2-\bar{K}_2)^2$$

c. $(\tau=1, \sigma=4)$: Here, proceeding as before, we have

$$K_4=-\tfrac{5}{32}(K_2-\bar{K}_2)^2, \qquad K_3=\tfrac{1}{2}(3K_2+\bar{K}_2)$$
$$\bar{K}_3=\tfrac{1}{2}(3\bar{K}_2+K_2), \qquad a_0=-4i/(K_3-\bar{K}_3)$$

d. $(\tau=4, \sigma=1)$: In this case we get

$$K_4=\tfrac{1}{128}(K_2-K_2)^2, \qquad K_3=\tfrac{1}{2}(3K_2+K_2)$$
$$\bar{K}_3=\tfrac{1}{2}(3\bar{K}_2+K_2), \qquad a_0=8i/(K_3-\bar{K}_3)$$

e. Now we consider (10) along with $K_3-K_3\neq K_2-\bar{K}_2$, i.e., for situations other than (28). From (30) we get

$$K_3-\bar{K}_3=(K_2-\bar{K}_2)\lambda, \qquad \lambda=2(2-\tau)/(\sigma-\tau-1)$$

whence

$$\bar{K}_3=\tfrac{1}{2}[(\lambda+2)\bar{K}_2-(\lambda-2)K_2]$$
$$K_4=\frac{1}{16(2-\tau)^2}[\tfrac{3}{2}\lambda^2-(2-\tau)\lambda(\lambda-2)](K_2-\bar{K}_2)^2$$
$$K_3=\tfrac{1}{2}[(\lambda+2)K_2-(\lambda-2)\bar{K}_2], \qquad \tau\neq 2 \tag{33}$$

So the resonances occur at $r=-1$, 0, 1, σ, τ $(K_2\neq\bar{K}_2,\ K_3\neq\bar{K}_3)$. We note that there are no other cases for (12)-(14).

3.2. Leading Orders Defined in (15)-(18)

a. When (28) holds, i.e., when $K_3-\bar{K}_3=K_2-\bar{K}_2$, along with $\sigma+\tau=5$, we have

$$a_0=\frac{4i(2p-1)}{K_2+\bar{K}_2+\bar{K}_3(p-1)-K_3p} \tag{34}$$

$$8(2p-1)^2K_4=\{p(p+1)[\bar{K}_2+\bar{K}_3(p-1)]+(p-1)(p-2) \times(K_2-K_3p)\}[K_2+\bar{K}_2+\bar{K}_3(p-1)-K_3p] \tag{35}$$

$$[(-32p^2+32p+16)+(6p-8)iK_3a_0+(6p+2)(iK_3a_0) -(4p-6)iK_2a_0-(4p+2)i\bar{K}_2a_0+a_0^2(K_3\bar{K}_3-K_2\bar{K}_3 -K_3\bar{K}_2)]=4\sigma(5-\sigma) \tag{36}$$

If we eliminate a_0 between (34) and (36), we get a fourth-degree equation in p and (35) is another equation of degree four in p, so the consistency between these two will give conditions for the occurrence of resonance positions at integer values of r (other than at $r=-1$) for a particular combination of K_2, K_3, $\bar{K}_2$, $\bar{K}_3$, and K_4.

The situation involving those particular values of K_i, $\bar{K}_i$ that lead to situations equivalent to that for which Newell (1979) obtained a Lax pair belong to this case. In that situation from (7) we have $K_2=-2$, $K_3=0$, $K_4=2$, $\bar{K}=-2$, and $\bar{K}_3=0$, for which (28) is satisfied. We observe that for those values of K_2, K_3, K_4, $\bar{K}_2$, and $\bar{K}_3$ from (35) and (34) [which are the same as (18) and (16), respectively] one has (i) $p=0$, $a_0=i$ or (ii) $p=1$, $a_0=-i$ [as in (19)]. With these values of p and a_0 one gets from (36) $\sigma=2$, 3, i.e., the resonance positions for these cases are

$$-1, 0, 1, 2, 3 \tag{37}$$

b. Another situation that arises when (28) does not hold is as follows. We have from (27) and (18)

$$a_0=\frac{2i[(\sigma+\tau)-5]}{(K_3-\bar{K}_3)-(K_2-\bar{K}_2)} \tag{38a}$$

$$a_0=\frac{4i(2p-1)}{K_2+\bar{K}_2+\bar{K}_3(p-1)-K_3 p} \tag{38b}$$

which when equated immediately lead to

$$p=\frac{2[(K_3-\bar{K}_3)-(K_2-\bar{K}_2)]+(K_2+\bar{K}_2-\bar{K}_3)(\sigma+\tau-5)}{4[(K_3-\bar{K}_3)-(K_2-\bar{K}_2)]+(K_3-\bar{K}_3)(\sigma+\tau-5)} \tag{38c}$$

But we have two other conditions,

$$\begin{aligned}8(2p-1)^2K_4=\{&p(p+1)[\bar{K}_2+\bar{K}_3(p-1)]+(p-1)(p-2)\\&\times(K_2-K_3 p)\}[K_2+\bar{K}_2+\bar{K}_3(p-1)-K_3 p]\end{aligned} \tag{39}$$

and equation (36) with the right-hand side replaced by $4\sigma\tau$ (σ, τ integer positive). So, on substitution of p from (38c) into these equations we get equations connecting K_2, K_3, $\bar{K}_2$, $\bar{K}_3$, and K_4, which are to be solved for various integral values of τ, σ.

4. SEARCH FOR ARBITRARY EXPANSION COEFFICIENTS AT THE RESONANCE POSITION

From (20) it is easy to check that, if

$$[K_2(r-1)+pK_3+2iK_4a_0]=M_1 \tag{40a}$$

$$[\bar{K}_2(r-1)-(p-1)K_3-2iK_4a_0]=M_2 \tag{40b}$$

$$[-2i(r-p)(r-p-1)-K_2a_0-K_3a_0(r-p)+iK_4a_0^2]=M_3 \tag{40c}$$

$$[-2i(r+p-1)(r+p-2)-\bar{K}_2a_0-\bar{K}_3a_0(r+p-1)-iK_4a_0^2]=M_4 \tag{40d}$$

then the compatibility condition is as follows:

(i) When

$$M_1\neq 0, \qquad M_2\neq 0, \qquad M_3\neq 0, \qquad M_4\neq 0 \tag{41a}$$

we have

$$M_3M_4X-2Sc_0(r-1)M_4Y-2Sb_0(r-1)M_3Z=0 \tag{41b}$$

with $n=r$.

(ii) When

$$M_1=0, \qquad M_2\neq 0, \qquad M_3=0, \qquad M_4\neq 0 \tag{42a}$$

we have

$$Y=0 \tag{42b}$$

with $n=r$.

(iii) When

$$M_1\neq 0, \qquad M_2=0, \qquad M_3\neq 0, \qquad M_4=0 \tag{43a}$$

we have

$$Z=0 \tag{43b}$$

with $n=r$.

However, (41)-(43) do not exhaust all possible situations, though it is true that they include a large number of situations and may facilitate further investigations.

In the following we consider some particular cases.

Case I. $K_2=-2$, $K_3=0$, $K_4=2$, $\bar{K}_2=-2$, $\bar{K}_3=0$. Here we discuss the situation given in (7), (19), and (37). We have seen that we have two branches, ($p=0$, $a_0=i$) and ($p=1$, $a_0=-i$). For both branches resonance positions are -1, 0, 1, 2, 3. The situations $r=-1$, 0 correspond to arbitrary $\phi(x,t)$ and the arbitrariness of one of b_0 and c_0 in (10). In the following we investigate the resonance positions $r=1$, 2, 3. For the branch given by $p=0$, $a=i$ we have the following

At $r=1$:

$$a_1=-\tfrac{1}{4}\dot{f} \tag{44a}$$

$$b_1=\text{arbitrary} \tag{44b}$$

$$c_1=-ic_0\dot{f}/4 \tag{44c}$$

subject to the compatibility condition $\dot{a}_0 = 0$, which is obtained due to the vanishing of the first row of the system matrix of (20), and is satisfied identically.

At $r = 2$: Here (41a) is satisfied and therefore the compatibility condition is given by (41b) with the particular values of K_i, $\bar{K}_i$, p, a_0, r concerned and is identically satisfied by the results for $j < 2$. The expansion coefficients are then given by

$$a_2 = \text{arbitrary} \tag{45a}$$

$$b_2 = -(i/32)(8\dot{b}_0 - 3i\dot{f}^2 b_0 + 48 b_0 a_2) \tag{45b}$$

$$c_2 = -(i/32)(8\dot{c}_0 + 3i\dot{f}^2 c_0 - 16 c_0 a_2 - 16iSc_0^2 b_1) \tag{45c}$$

At $r = 3$: Here (43a) is satisfied and hence the compatibility condition is given by (43b) with the particular values of K_i, $\bar{K}_i$, p, a_0, r concerned and is identically satisfied by the results for $j < 3$. The expansion coefficients are then given by

$$a_3 = (1/2\dot{f})[-4Sc_0 b_3 - 4Sb_0 c_3 + \dot{a}_2 - 4S(b_1 c_2 + b_2 c_1)] \tag{46a}$$

$$\begin{aligned} b_3 = (1/12i)[&8b_0 a_3 + \dot{b}_1 + (6b_1 - 4ia_1 b_0)a_2 + 4a_1 b_2 \\ &+ 2iSb_0^2 c_2 - 2ia_1^2 b_1 + 2iSb_1^2 c_0 + 4iSb_0 b_1 c_1] \end{aligned} \tag{46b}$$

$$c_3 = \text{arbitrary} \tag{46c}$$

Similarly, for the branch given by $p = 1$, $a_0 = -i$, we have the following:
At $r = 1$:

$$a_1 = -\dot{f}/4 \tag{47a}$$

$$b_1 = ib_0\dot{f}/4 \tag{47b}$$

$$c_1 = \text{arbitrary} \tag{47c}$$

subject to the compatibility condition $\dot{a}_0 = 0$, which is obtained due to the vanishing of the first row of the system matrix of (20), and is satisfied identically.

At $r = 2$: Here also (41a) is satisfied and therefore the compatibility condition is given by (41b) with the particular values of K_i, $\bar{K}_i$, p, a_0 and r concerned and is identically satisfied by the results for $j < 2$. The expansion coefficients are then given by

$$a_2 = \text{arbitrary} \tag{48a}$$

$$b_2 = (i/32)(8\dot{b}_0 - 3i\dot{f}^2 c_0 - 16 b_0 a_2 + 16iSb_0^2 c_1) \tag{48b}$$

$$c_2 = (i/32)(8\dot{c}_0 + 3i\dot{f}^2 c_0 + 48 c_0 a_2) \tag{48c}$$

At $r=3$: Here (42a) is satisfied and therefore the compatibility condition is given by (42b) with the particular values of K_i, $\bar{K}_i$, p, a_0, and r concerned and is identically satisfied by the results for $j<3$. The expansion coefficients are then given by

$$a_3=(1/2\dot{f})[-4Sc_0b_3-4Sb_0c_3+\dot{a}_2-4S(b_1c_2+b_2c_1)] \tag{49a}$$

$$b_3=\text{arbitrary} \tag{49b}$$

$$\begin{aligned} c_3=(1/-12i)[&8c_0a_3+\dot{c}_1+(6c_1+4ia_1c_0)a_2+4a_1c_2 \\ &-2iSc_0^2b_2+2ia_1^2c_1-2iSc_1^2b_0-4Sc_0b_1c_1] \end{aligned} \tag{49c}$$

Thus, for both branches, (9) represents a general solution (8) containing a maximum number of arbitrary functions allowed at the resonances satisfying the Cauchy-Kovalevskaya theorem and thus all the criteria in connection with (1) and (2) are satisfied and the system passes the Painlevé test in the sense of Weiss *et al.* (1983; Weiss, 1985a,b). According to the conjecture of Weiss *et al.* (1983; Weiss, 1985a,b)), the system should be integrable in this situation. We have discussed in the introduction that the Lax pair exists (Newell, 1979) for this system in this situation.

Case II. $p=\frac{1}{2}$, $K_2=\bar{K}_2$, $K_3=\bar{K}_3$, $2K_2=K_3$, and $r=-1, 0, 1, 2, 3$, which is a special situation (when $\tau=2$, $\sigma=3$) of those considered before in (30):

(i) $r=-1$ corresponds to an arbitrary $\phi=x-f(t)$.
(ii) $r=0$ corresponds to the arbitrariness of either b_0 or c_0.
(iii) For $r=+1$, we get

$$a_1=\text{arbitrary} \tag{50a}$$

$$b_1=\frac{-(\dot{f}-2ia_0\dot{f})b_0+(K_3+4iK_4a_0)b_0a_1}{2(2i+K_3a_0)} \tag{50b}$$

$$c_1=\frac{-(\dot{f}+2ia_0\dot{f})c_0+(K_3-4iK_4a_0)c_0a_1}{2(-2i+K_3a_0)} \tag{50c}$$

(iv) For $r=2$ we get the following relation between the coefficients. Here (41a) is satisfied, so that the compatibility condition is given by (41b) with the particular values of K_i, $\bar{K}_i$, p, a_0 and r concerned, leading to

$$\begin{aligned} &\dot{a}_1-2Sb_1c_1+\frac{Sc_0}{K_3a_0}\left[\dot{b}_0-\frac{b_1\dot{f}}{2}+\frac{K_3a_1b_1}{2}-2iK_4a_0a_1b_1+4iSb_1b_0c_0+2iSb_0^2c_1\right] \\ &+\frac{Sb_0}{K_3a_0}\left[\dot{c}_0-\frac{c_1\dot{f}}{2}+\frac{K_3a_1b_1}{2}+2iK_4a_0a_1c_1-4iSc_1c_0b_0-2iSc_0^2b_1\right]=0 \end{aligned} \tag{51}$$

which is a differential equation in a_1.

So a_1 is fixed and we cannot meet the requirement of the Cauchy-Kovalevskaya theorem as utilized in the formalism of Weiss *et al.* (1983; Weiss, 1985a,b). So the system (8) in this situation does not pass the Painlevé test in the sense of Weiss *et al.* (1983; Weiss, 1985a,b).

Case III. Here

$$p=\tfrac{1}{2},\qquad K_4=\tfrac{1}{32}(K_2-\bar{K}_2)^2,\qquad K_3=\tfrac{1}{2}(3K_2+\bar{K}_2)$$

$$\bar{K}_3=\tfrac{1}{2}(3\bar{K}_2+K_2),\qquad a_0=\frac{4i}{K_3-\bar{K}_3},\qquad K_3\neq\bar{K}_3,\qquad K_2\neq\bar{K}_2$$

There is resonance at $r=-1$, 0, 1, 2, 3. This situation is treated in (31). Here also a_1 becomes fixed due to a compatibility condition at $r=2$, which is similar to (51). Thus, for the same reason as stated in case II, the system in this situation does not pass the Painlevé test in the sense of Weiss *et al.* (1983; Weiss, 1985a,b).

Case IV. Here

$$\text{(i)}\ p=\tfrac{1}{2},\qquad a_0=-8i/(K_3-\bar{K}_3),\qquad K_3=\tfrac{1}{2}(3K_2+\bar{K}_2)$$

$$\bar{K}_3=\tfrac{1}{2}(K_2+3\bar{K}_2),\qquad K_4=\tfrac{7}{128}(K_2-\bar{K}_2)^2,\qquad K_3\neq K_3,\qquad K_2\neq\bar{K}_2$$

$$\text{(ii)}\ p=\tfrac{1}{2},\qquad a_0=12i/(K_3-\bar{K}_3),\qquad K_3=\tfrac{1}{2}(3K_2+\bar{K}_2)$$

$$\bar{K}_3=\tfrac{1}{2}(K_2+3\bar{K}_2),\qquad K_4=-\tfrac{3}{288}(K_2-\bar{K}_2)^2\qquad K_3\neq K_3,\qquad K_2\neq\bar{K}_2$$

These situations were discussed in the context of resonance determination with the help of (30) with σ, $\tau=0$, 5. For both situations the resonances are -1, 0, 0, 1, 5. So we have a double resonance at $r=0$, while a_0 is fixed and only one of b_0 and c_0 is permitted to be arbitrary in (10c). So, for reasons as stated in case II, we may infer that the system (8) does not pass the Painlevé test in the sense of Weiss *et al.* (1983; Weiss, 1985a,b).

Case V. Here

$$\text{(i)}\ p=\tfrac{1}{2},\qquad a_0=-4i/(K_3-\bar{K}_3),\qquad K_3=\tfrac{1}{2}(3K_2+\bar{K}_2)$$

$$\bar{K}_3=\tfrac{1}{2}(3\bar{K}_2+K_2),\qquad K_4=\tfrac{5}{32}(K_2-\bar{K}_2)^2,\qquad K_3\neq K_3,\qquad K_2\neq\bar{K}_2$$

$$\text{(ii)}\ p=\tfrac{1}{2},\qquad a_0=8i/(K_3-\bar{K}_3),\qquad K_3=\tfrac{1}{2}(3K_2+\bar{K}_2)$$

$$\bar{K}_3=(3\bar{K}_2+K_2),\qquad K_4=-\tfrac{1}{128}(K_2-\bar{K}_2)^2,\qquad K_3\neq\bar{K}_3,\qquad K_2\neq\bar{K}_2$$

These situations were discussed in the context of resonance determination with the help of (30) with σ, $\tau=1$, 4. For both the situations the resonances are -1, 0, 1, 1, 4. So we have a double resonance at $r=1$. We can check that if we do not further restrict the values of K_i, the double resonance at

Table I. Results

Parameter K_i, p, a_0 values	Resonance position	Arbitrariness in the series	Remarks
I. $K_2=-2$, $K_3=0$, $K_4=2$ $\bar{K}_2=-2$, $\bar{K}_3=0$ For these K_i, Newell (1979) obtained a Lax pair; here we have two situations:			
(i) $p=0$, $a_0=i$	$r=-1, 0, 1, 2, 3$	$\phi(x, t)$, one of b_0 and c_0, b_1, a_2, and c_3 can be kept arbitrary	The system (8) passes the Painlevé test for integrability in the sense of Weiss *et al.* (1983; Weiss, 1985a,b)
(ii) $p=1$, $a_0=-i$	$r=-1, 0, 1, 2, 3$	$\phi(x, t)$, one of b_0 and c_0, c_1, a_2, and b_3 can be kept arbitrary	As above
II. $p=\frac{1}{2}$, $K_3=\bar{K}_3$, $K_2=\bar{K}_2$, $a_0^2=-\dfrac{3}{2K_4}$	$r=-1, 0, 1, 2, 3$	One of b_0 and c_0 can be arbitrary, a_1 arbitrary, but from compatibility at $n=2$, a_1 is fixed	The number of arbitrary functions in the series expansions for A, B, C is less than the maximum number of arbitrary functions allowed at the resonances and thus we have a violation of the conjecture of Weiss *et al.* (1983; Weiss, 1985a, b) regarding integrability
III. $p=\frac{1}{2}$, $K_3=\frac{1}{2}(3K_2+\bar{K}_2)$, $\bar{K}_3=\frac{1}{2}(3\bar{K}_2+K_2)$, $K_4=\frac{1}{32}(K_2-\bar{K}_2)^2$, $a_0=4i/(K_2-\bar{K}_2)$, $K_3\neq\bar{K}_3$, $K_2\neq\bar{K}_2$	$r=-1, 0, 1, 2, 3$	As above	As above

IV. (i) $p=\frac{1}{2}$, $K_3=\frac{1}{2}(3K_2+\bar{K}_2)$, $\bar{K}_3=\frac{1}{2}(3\bar{K}_2+K_2)$, $K_4=\frac{7}{128}(K_2-\bar{K}_2)^2$, $a_0=-8i/(K_3-\bar{K}_3)$, $K_3\neq\bar{K}_3$, $K_2\neq\bar{K}_2$	$r=-1, 0, 0, 1, 5$	a_0 fixed, but one of b_0 and c_0 can be arbitrary	The number of arbitrary functions in the series expansions for A, B, C is less than the maximum number of arbitrary functions allowed at the resonances, and thus we have a violation of the conjecture of Weiss *et al.* (1983; Weiss, 1985a,b) regarding integrability
(ii) $p=\frac{1}{2}$, $K_3=\frac{1}{2}(3K_2+\bar{K}_2)$ $\bar{K}_3=\frac{1}{2}(3\bar{K}_2+K_2)$, $K_4=-\frac{3}{288}(K_2-\bar{K}_2)^2$, $a_0=12i/(K_2-\bar{K}_2)$, $K_3\neq\bar{K}_3$, $\bar{K}_2\neq K_2$	$r=-1, 0, 0, 1, 5$	As above	As above
V. (i) $p=\frac{1}{2}$, $\bar{K}_3=\frac{1}{2}(3\bar{K}_2+K_2)$ $K_3=\frac{1}{2}(3K_2+\bar{K}_2)$, $K_4=\frac{5}{32}(K_2-\bar{K}_2)^2$, $a_0=-4i/(K_2-\bar{K}_2)$, $K_3\neq\bar{K}_3$, $K_2\neq\bar{K}_2$	$r=-1, 0, 1, 1, 4$	One of b_0 and c_0 can be arbitrary at $n=0$; though there is a double resonance at $r=1$, one among a_1, b_1, c_1 can be arbitrary; this result was checked without imposing further restrictions on K_i	The number of arbitrary functions in the series expansions for A, B, C is less than the maximum number of arbitrary functions allowed at the resonances, and thus we have a violation of the conjecture of Weiss *et al.* (1983; Weiss, 1985a,b) regarding integrability
(ii) $p=\frac{1}{2}$, $K_3=\frac{1}{2}(3K_2+\bar{K}_2)$, $\bar{K}_3=\frac{1}{2}(3\bar{K}_2+K_2)$, $K_4=-\frac{1}{128}(K_2-\bar{K}_2)^2$, $a_0=8i/(K_2-\bar{K}_2)$, $K_3\neq K_3$, $K_2\neq\bar{K}_2$	$r=-1, 0, 1, 1, 4$	As above	As above

$r=1$ does not seem to be satisfied, and thus, for reasons stated in case II, here also we may say that the system (8) does not pass the Painlevé test in the sense of Weiss *et al.* (1983; Weiss, 1985a,b).

5. CONCLUSION

We have attempted to apply the conjecture of Weiss *et al.* (1983; Weiss, 1985a,b) regarding integrability to the long-wave, short-wave equations. The results are summarized in Table I. The results do not exhaust all possible combinations of K_i. However, we could do this up to the leading order analysis and the calculation of resonance positions, where we obtained several restrictions on K_i, $\bar{K}_i$. Unless one knows the resonance positions exactly, one cannot check the arbitrariness at the resonance positions. For this reason we had to restrict ourselves to some particular values (or combinations) of K_i, $\bar{K}_i$ obeying the conditions already imposed in the previous steps. We make the following observations.

1. The system (8) with $K_2=-2$, $K_3=0$, $K_4=2$, $\bar{K}_2=-2$, $\bar{K}_3=0$ is equivalent to the long-wave, short-wave equations (4) with (6), for which Newell (1979) obtained a Lax pair, and with those K_i the system (8) passes the Painlevé test for integrability in the sense of Weiss *et al.* (1983; Weiss, 1985a,b).

2. There are number of other combinations of K_i, $\bar{K}_i$ for which the system (8) does not pass the Painlevé test for integrability in the sense of Weiss *et al.* (1983; Weiss, 1985a,b). It is interesting to note that for equivalent situations of the long-wave, short-wave equations (4), Newell (1979) did not report the existence of a Lax pair.

At this point it might not be out of place to note that our Painlevé analysis cannot be used to deduce the Lax pair because there is still no concrete method to deduce a Lax pair for the coupled nonlinear equations.

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